Assessing from a CGI Perspective

Cheryl Ann Lubinski*, JoAnn Cady, Heather Bailey

Abstract
This is the story of Heather, a K–1 teacher who wanted to develop an assessment instrument to determine the strategies her students used to solve problems involving all four basic operations—addition, subtraction, multiplication, and division. Heather asked her mathematics education mentor Cheryl to help her.

Keywords: Cognitively Guided Instruction, early childhood, conceptual understanding, mathematics education

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1. Introduction

This article emerged from our ongoing collaboration as teachers and mathematics educators who use Cognitively Guided Instruction (CGI) in our classrooms and in staff development sessions with preservice and in-service elementary teachers. CGI is an established research-based program that originated at the University of Wisconsin–Madison designed, in part, to help teachers address diverse learners. The premise underlying CGI is that if teachers learn how children come to understand mathematical ideas, they can provide better instruction for the children in their classes. According to the National Research Council:

CGI is a professional development program for teachers that focuses on helping them construct explicit models of the development of children’s mathematical thinking in well-defined content domains. No instructional materials or specifications for practice are provided in CGI; teachers develop their own instructional materials and practices from watching and listening to their students solve problems. (National Research Council 2001, 400)

Teachers in CGI classrooms carefully create and assign relevant problems in order to assess their students’ learning strategies, and then they plan instruction to develop those strategies along with their students’ mathematical understanding. The primary goal of this article is to provide teachers with an instrument to assess their primary-age students’ learning strategies.

In addition, we connect CGI to the Common Core State Standards of Mathematical Practice (CCSS-MP), as do many articles about CGI. For example, the existing literature on CGI addresses the need for teachers to:

- focus on problem solving by making sense of problems and persevere in solving them (Hoosain & Chance, 2004);
- prepare students for problem solving by having them use appropriate tools strategically (Holden, 2007);
- use problems and numbers that are appropriate to students’ developmental levels to reason abstractly and quantitatively (Jacobs & Ambrose, 2008);
- have students model with mathematics and attend to precision by using mathematical terms appropriately (Carpenter et al., 2003);
- listen to and develop students’ strategies by having them construct viable arguments (Hiebert et al., 1997);

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Illinois Mathematics Teacher
• record symbolic representations that connect to students' thinking as they look for and make use of structure (Hoosain & Chance, 2004); and

• pose questions that assist students to verbalize their reasoning process as they look for and express regularity in repeated reasoning (Jacobs & Ambrose, 2008).

CGI helps us to measure what each student knows to inform how instruction should be structured to enable a particular student to learn (Chambers & Lacampagne, 1994). The authors of the articles cited above recognize that CGI teachers encourage their students to develop the mathematical practices set forth in the Common Core State Standards. However, no article specifically outlines an individual assessment approach to determine the developmental level reflected by a student's strategies when teaching from a CGI perspective. In this article, we aim to fill this void by providing an assessment instrument that helps to determine a student’s developmental level as well as to address a student’s level of achievement in regard to the Common Core.

2. Background of the Authors

Cheryl and JoAnn are university faculty members with prior elementary teaching experience. They are currently working with elementary preservice and inservice teachers on implementing CGI and Common Core standards and practices in classrooms. Heather was one of the elementary teachers participating in a series of CGI professional development workshops led by Cheryl. She is a K–1 teacher who loops (teaches the same students for two years—in kindergarten and in first grade) and is new to the ideas of CGI. Fifty-two percent of students in her school are on free or reduced lunch. After these workshops, Cheryl and Heather established a two-year “co-mentoring” relationship in which Cheryl routinely co-taught in Heather’s classroom in order to better develop our own learning about students’ thinking. As is typical in a CGI classroom, we gave students a variety of problem types to solve, discussed their strategies between us, and talked about implications for lesson planning that incorporated differentiated instruction. After one of our collaboration sessions, Heather suggested we develop an assessment tool in order to obtain a more accurate description of her individual student’s strategies as well as their developmental level when solving problems involving all four operations with whole numbers. The remainder of this article describes the assessment tool that we developed.

3. Background Knowledge for Understanding Our Assessment Tool

In order to understand our assessment tool, it is important for teachers to have knowledge about Problem Types and Solution Strategies, as these terms are used in the CGI literature. We will provide a brief explanation here. For a more detailed description see Carpenter et al. (2015).

3.1. Problem Types

Word problems frequently found in textbooks for the K–3 teacher are typically limited in their structure, as indicated by the location of the unknown, as in the following word problem:

Laura has 5 dolls and she gets 4 more. How many dolls does she have now?

The structure of this word problem is symbolically represented as:

\[ 5 + 4 = n, \]

where \( n \) represents the unknown.

Compare this with the alternatives below:

Laura has 5 dolls. How many more does she need to have 9?

The structure of this word problem is symbolically represented as

\[ 5 + n = 9. \]

Laura has some dolls. She gets 4 more. Now she has 9 dolls. How many dolls did Laura have to start?

Illinois Mathematics Teacher
The structure of this word problem is symbolically represented as:

\[ n + 4 = 9. \]

In a CGI classroom, word problems vary in structure, and teachers initially begin to use those from the problem types chart found in Appendix A. Problems in this chart are categorized as *Join*, *Separate*, *Part-Part-Whole*, *Compare*, *Grouping*, and *Partitioning*. Each of these six categories has variations based on the location of the unknown. For example, in the Join row we have *Result Unknown*, *Change Unknown*, and *Start Unknown*, abbreviated as JRU, JCU, and JSU. We see that the unknowns are located at the end, in the middle, and at the beginning, respectively. The Separate problems follow a similar structure. The Part-Part-Whole problems have either the whole or the part unknown. The Compare problems have a difference, a compare quantity, or referent unknown. Grouping problems have the total number unknown. Partitioning problems are of two types, either the number of groups is unknown (*Measurement Division*) or the number in a group is unknown (*Partitive Division*).

### 3.2. Strategies

CGI research provides compelling evidence that even young students have strategies to solve word problems involving whole numbers with all four operations. Joining, Separating, Grouping, and Partitioning problems involve actions; thus the *Direct Modeler* (defined in Table 1) can solve these more easily than problems that don’t have actions, such as the Compare and Part-Part-Whole problems. The Start Unknown problems are the exception because Direct Modelers have no number with which to start and can only solve these problem types with a *Trial and Error* strategy.

It is important for teachers not to connect the problem type category to an operation. How students solve a problem indicates how they think about the situation and provides developmental insight into their strategy level. For example, consider the JCU problem below:

<table>
<thead>
<tr>
<th>Direct Modeling</th>
<th>Student represents all of the numbers in a word problem with counters, fingers, or pictures, in the order that the numbers are given.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting</td>
<td>Student relies on a counting strategy rather than modeling each number.</td>
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<tr>
<td>Derived Facts</td>
<td>Student uses an already known fact to determine an unknown fact.</td>
</tr>
<tr>
<td>Number Facts</td>
<td>Student uses any fact about one of the four operations with the numbers 0–12.</td>
</tr>
</tbody>
</table>

Table 1: Problem-solving strategy categories

*Anyae has 5 bows, and Gloria gives her some more. Now Anyae has 11 bows. How many bows did Gloria give Anyae?*

A student responding to this problem could:

- count out five cubes and join cubes to the five cubes until the number eleven is reached. Then the student would count the cubes joined and say, “six.” (Student Directly Models.)
- say, “five,” count on her fingers, “six, seven, eight, nine, ten, eleven,” count the fingers she used, and say, “six.” (Student uses a Counting strategy.)
- say, “eleven, because five plus five is ten and one more is eleven, and five plus one is six.” (Student uses a Derived Fact.)
- start by saying, “eleven,” and count back until the number five is reached, using fingers or counters to count back, then count how many were taken away and say, “six.” (Student uses a Counting strategy.)
- say, “six, because I just know five plus six is eleven.” (Student uses recall of Number Facts.)

Thus either a joining or separating action can be connected to a solution strategy. All of the above strategies provide a correct answer, but how the student solves the problem provides information about the student’s strategy level. In Appendix B, we provide Individual Assessment Sheets for documenting both the strategies a student uses.
and the mathematical practices a student demonstrates.

3.3. The Strategies a Student Uses

On the first Individual Assessment Sheet in Appendix B, there is a list of strategies a student could use. Each of the strategies listed in Appendix B are explained in more detail in Appendix C. The strategies students use provide insight to their developmental level, as described in the CGI literature. Initially, a student Directly Models solution strategies, using one of the four Direct Modeling strategies listed. Then a student moves to using one of the eight Counting strategies. Finally, students will routinely use Derived Facts or known Number Facts. It is important to note that even Direct Modelers and Counters use some Derived Facts and Number Facts as they solve problems.

In this article, we refer to Beginning Problem Solvers, Developing Problem Solvers, and Mature Problem Solvers. We provide a brief description of each in the next paragraphs.

Beginning Problem Solvers are students who can successfully find a solution to a problem using a Direct Modeling solution strategy. Some need to hear the problem one step at a time. Some need to work on counting with one-to-one correspondence. As they practice “telling the story” (saying the word problem in their own words), they can eventually recount the complete problem and attempt to solve it. They have the most success with JRU, JCU, PPWwu, SCU, and SRU problems for the Addition and Subtraction problem types. They can often solve Grouping and Partitioning problems because these problem types have actions students can easily model.

Developing Problem Solvers use more abstract strategies than Beginning Problem Solvers. They do not need to represent all numbers in the problem with counters and often use Counting strategies as well as Derived Facts and Number Facts. They plan ahead when solving problems, i.e., they think before doing.

Mature Problem Solvers use more abstract strategies routinely, often in their heads using Derived Facts and Number Facts. Additionally they can solve all of the problem types using strategies that sometimes rely on inverse number relationships and the Commutative Property. Their flexible use of strategies helps them to be successful with operations involving multi-digit numbers as well as with JSU and SSU problems—problems that Beginning and Developing Problem Solvers can only solve with a Trial and Error strategy.

3.4. The Mathematical Practices a Student Demonstrates

The Common Core State Standards recommends that a teacher help students develop mathematical practices (Council of Chief State School Officers, 2011). The resources cited in this report go into detail about what can be developed for each practice. We have included an assessment sheet for the Mathematical Practices in Appendix B so that teachers have a form on which to take notes about an individual student’s progress with these practices.

4. One Option for Assessing

Using the flow chart in Appendix D, we can assess an individual student’s strategy level, which is developmental. The flow chart provides one example of how to assess. A teacher may choose a different order in which to present the problem types depending on the teacher’s objective for assessment. During assessment, the teacher chooses number sets that will provide information on the student’s most typical strategies. The teacher observes and takes notes on how the student is solving the problem, using the Individual Assessment Sheets from Appendix B and referring to Appendix C to help with strategy identification. It is recommended that teachers give individual students between five and seven different problem types in order to fully describe how that student characteristically solves problems (Fennema et al., 2000).

To begin an assessment, we recommend using a JCU problem because this problem type can be solved at all developmental levels with Direct Modeling, Counting, Derived Facts, or known Number Facts. Additionally, if a student solves a
Assessing from a CGI Perspective

Figure 1: A Developing Problem Solver uses a Counting strategy to solve his problem: If a student has 51 trucks, how many more does he need to have 54 trucks? The teacher decided to list the numerals so the student can focus on the strategy.

JCU problem the student usually can solve a JRU or SRU problem.

Number suggestions are provided for each of the problem types in Appendix A so that teachers can make decisions based on their objectives for assessment. For example, for JCU, the numbers in the problem could be 5 and 8 (the first numerals in each of the two number sets) to assess how a student determines more; 37 and 47 to assess how a student determines a group of 10 more; or 89 and 109 to assess how a student determines 20 more and bridges the number 100, i.e., “does the student Count On by ones or by groups of tens?” Using groups of tens indicates the student uses knowledge of place value to solve problems involving multi-digit numbers.

5. Implementing a Lesson Based on the Assessment Results

Using the assessment instrument we developed, Heather assesses her students’ strategies, both individually and during group instruction; then she plans for instruction using her findings from the assessments. During instruction she notes which students need probes to encourage more abstract strategies, such as going from Direct Modeling strategies to Counting strategies, or from Counting strategies to Derived Fact strategies, and which students need practice at their current developmental level. Every lesson provides Heather with opportunities to refine her assessments of individuals.

Heather’s purpose for instruction is to develop meaningful strategies that the students can use with understanding. She carefully plans probes, both prior to and during instruction, that encourage her students to use their most sophisticated strategies based on the results of her assessment. For the students who need to Directly Model, she encourages pictures as a representation or has the students tell her about their thinking using manipulatives they select to help them explain. For students who use Counting strategies, she expects them to use those strategies rather than draw pictures, thus encouraging them to reason more maturely or abstractly about quantities. To encourage the use of Derived Facts, she might ask students what they know about “five and five” to help them to think about “five and six,” thus encouraging them to reason quantitatively about number relationships.

An example of how this might look in a classroom is given using the Dalmatian Puppy Problem presented to Heather’s entire class:

There are four Dalmatian puppies. Each puppy has five large spots. How many large spots are there?

This problem was chosen because it is appropriate for a diverse group of problem solvers. It can be Directly Modeled or solved using more mature strategies such as Skip Counting or Number Facts. After the problem is presented, many students use their fingers to solve it. Anyae, however, appears to be stuck. Anyae is a student who needs to Directly Model the structure of a mathematics problem and she struggles to explain her reasoning process. To encourage Anyae to persist to solve the problem, the teacher uses helpful probes so Anyae can, “make sense of the problem and persevere” (Common Core State Standards Initiative, 2010, CCSS.MP.1) as she solves it at her developmental level of Direct Modeling. Following is an
episode that is representative of how CGI’s philosophy encourages mathematical thinking involving Anyae’s work with that of the rest of the class:

Teacher: There are four Dalmatian puppies. Each puppy has 5 large spots. How many large spots are there? [Pause]

Teacher to Anyae: Tell me the story.

Anyae: There are four puppies. . . .

Teacher: What do they have?

Anyae: I don’t remember.

Teacher: They each have five large spots. I want you to figure out how many large spots there are. Tell me the whole story please.

Anyae: There are four puppies. They have five large spots. How many large spots are there?

Teacher: What could you do to figure this out?

Anyae shrugs.

Teacher: Draw a picture of the puppies and their spots on the white board.

(Note: It is appropriate for a teacher to suggest what a student could do or think about if the teacher has assessed the student’s developmental level and knows what the student needs to think about or do to solve a problem. The teacher has assessed that Anyae needs to Directly Model the problem situation.)

Teacher (while Anyae is drawing): Nora, what did you do?

Nora: I got 19. I counted, “1, 2, 3, 4, 5…” [Nora extends one finger to keep track of the number of groups of five she has counted] “6, 7, 8, 9, 10…” [two fingers] “11, 12, 13, 14, 15…” [three fingers] “16, 17, 18, 19” [four fingers].

Teacher: Show me the last part again. [As Nora repeats the last part, she self corrects, “16, 17, 18, 19, 20. It’s 20.” The teacher then asks Luke what he did.]


The teacher goes back to Anyae’s picture and asks her:

- How many puppies do you have?
- How many large spots are on each?
- Can you count them for the class?

Anyae answers each question and then counts the spots by ones and gets twenty. During this question-and-answer time, Heather is assessing how well Anyae makes sense of the problem.

Heather then decides to see if Nora can connect her strategy to Luke’s; thus encouraging Nora to reason more abstractly about quantities.

Heather: Nora, why do you go “1, 2, 3, 4, 5,” and then pause?

Nora: Because there are five large spots on the first puppy.

Heather: Do you know why Luke went, “5, 10, 15, 20?”

Through discussion, Nora realizes that she doesn’t have to count by ones and pause after every fifth number, instead, she can just say every fifth number, i.e., “5, 10, 15, 20.” (Nora still uses fingers to represent each group of five.) It is important to note here that Heather needs to go back to what Nora says to make connections to what Nora could learn to do. Heather makes a note to check to see if Nora uses this Skip Count strategy for a similar problem in the future.

Assessing from a CGI Perspective

realizes that four fives are twenty, and he can represent his thinking by the expression $4 \times 5 = 20$. Making connections from Luke’s thinking to symbols encourages him to model his mathematical thinking with symbols. Through discussion with the teacher, he described his understanding of the problem, and she helped him to connect this understanding to the structure of the problem as illustrated with having a multiplication representation. Additionally, Heather could have asked the class:

- How do all three solution strategies compare?
- How do Anyae’s groups of five spots connect to Luke’s skip counting strategy?
- How does Anyae’s groups of five spots connect to Nora’s counting the individual spots using her fingers to keep track of the number of dogs?

These teacher moves allow students to develop a deeper understanding of the mathematics.

Both Nora and Luke have been assessed as students who use Counting strategies routinely. Additionally, they are part of a group of students who are the most mature problem solvers in the class. Heather has encouraged them to use what they know about addition and subtraction facts to solve problems involving multi-digit numbers. This reflects CCSS Mathematical Practice 8, “Look for and express regularity in repeated reasoning,” or, using repeated reasoning that can encourage students to reason more abstractly and quantitatively (Common Core State Standards Initiative, 2010). With this one problem, Heather has individualized instruction in a whole class setting as she encourages CCSS Mathematical Practices.

6. Heather’s Most Mature Problem Solvers

An example of Heather’s instruction with her most mature problem solvers is illustrated when she presented the following problem (which initially appeared in CGI materials for teacher leaders):

Ilinois Mathematics Teacher

The baby elephant at the zoo weighed 488 pounds. During the summer she gained 145 pounds. How much does she weigh now?

Several of the first graders thought about it and then shared their thinking. Heather has to listen carefully and ask questions in order to understand the strategies her students use.

- Jude said, “400 plus 100 is 500 plus 80 is 580 plus 20 from the 40 [thinking $45 = 20 + 25$]. You have 600 plus 25 plus 5 plus 3 [$8 = 5 + 3$], which is 633.”

- Luke shared a different strategy: “488 plus 100 is 588 plus 12 is 600 plus 33 is 633.” Heather asked Luke, “Where did the 12 come from?” Luke replied, “It was inside the 45 [$45 = 12 + 33$].”

- Whitney provides yet another strategy: “400 plus 100 is 500 and 80 plus 20 is 100 and 500 plus 100 is 600. 600 plus 20 plus 8 is 628 plus 5 [she counted up by ones using her fingers] is 633.”

Heather’s most advanced Mature Problem Solvers were able to achieve success because Heather carefully assesses what each student in her class can do and plans individual questions for them as she teaches in a whole class or small group setting. In this way she truly attends to the diversity of her students’ needs. As a consequence, each student in her class is challenged to reason and problem-solve at a level that is in alignment with their developmental levels, as indicated by the strategies they use most often.

7. Discussion of the Assessment Process

Teachers who have knowledge of problem types and students’ strategies can choose problems and numbers for their students that encourage more sophisticated strategies and increase mathematical understandings. Additionally, using the lists in Appendix B, teachers can keep track of students’ progress throughout the year. Progress is determined by
how students are developing their strategies to more abstract levels. Assessment sheets could be kept in a folder for individual students along with student work that supports the teacher’s comments. These assessment sheets can be used for both planning throughout the year and reporting student progress during conference time. These sheets are a record of what the students need in order to be successful as well as of their individual goals for the year.

When teachers assess students’ strategies while solving word problems, they can plan questions to ask during instruction, as Heather does, that provide opportunities for diverse problem solvers to progress from Directly Modeling, to Counting, to Derived Facts, and finally to Number Facts. Most students know some Number Facts, and encouraging students who are ready to derive new Number Facts from those that they already know can help them attain fact mastery. Therefore, recording known facts involving all four operations on a student’s individual assessment sheet as well as realizing what facts students can derive could aid teachers when planning individual and group instruction.

Teachers can help students move through the developmental levels by selecting appropriate problem types and numbers that help students to develop more abstract strategies over time. Knowing how a student solves problems is important for planning whole-group lessons. For example, during instruction a teacher who has knowledge of the students’ strategies can plan to ask individual students questions appropriate to each of their developmental levels.

We have used all of these problem types in our work in pre-K–3 classrooms. The introduction of multiplication and division problem types is beneficial to understanding place value concepts. Thus work with these problem types should begin early, ideally in kindergarten. Beginning problem solvers need to practice their strategies until they can plan ahead and solve problems involving all four operations with whole numbers. Mature problem solvers need challenging problems to further develop place-value concepts and relational thinking. All problem solvers need practice with strategies that encourage more abstract reasoning.

8. Reflecting on Our Collaborations

As we reflect on our work, we realize that using CGI and continually focusing on assessing students’ mathematical thinking not only informs our instructional decisions, but also helps us to develop the Mathematical Practices described in the CCSS-M documents. Additionally, the co-mentoring relationship I had with Heather helped Heather attend to precision with her students (Common Core State Standards Initiative, 2010, CCSS.MP.6), when she asks probing questions and uses appropriate mathematical terminology and mathematically correct representations. Our collaborative work was successful because we had a desire to learn and to develop within each of our areas of expertise—be it conducting staff development sessions, working with preservice teachers, or learning about K–3 students’ strategy development. Our assessment instrument contributed to our success.

References


Illinois Mathematics Teacher
Assessing from a CGI Perspective


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### Problem Types Chart

<table>
<thead>
<tr>
<th>Problem Type</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Join</strong></td>
<td><em>(Result Unknown, JRU)</em> <em>(Student's name) has (2, 18, or 98) jelly beans and I give you (5, 10, or 10) more. Now how many do you have?</em></td>
</tr>
<tr>
<td></td>
<td><em>(Change Unknown, JCU)</em> <em>(Student's name) has (5, 37, or 89) M&amp;Ms. I give her some more and now she has (8, 47, or 109). How many did I give her?</em></td>
</tr>
<tr>
<td></td>
<td><em>(Start Unknown, JSU)</em> <em>(Student's name) has some trucks. I give you (3, 10, or 35) more. Now you have (7, 27, or 90) trucks. How many trucks did you have to start?</em></td>
</tr>
<tr>
<td><strong>Separate</strong></td>
<td><em>(Result Unknown, SRU)</em> <em>(Student's name) has (5, 21, 84, or 101) cookies and I take (2, 10, 20, or 35). Now how many does he have?</em></td>
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<tr>
<td></td>
<td><em>(Change Unknown, SCU)</em> <em>(Student's name) has (9, 30, or 102) pretzels. I take some and now you have (7, 20, or 80). How many did I take?</em></td>
</tr>
<tr>
<td></td>
<td><em>(Start Unknown, SSU)</em> <em>(Student's name) has some Legos. I take (3, 10, or 35) from you. Now you have (4, 17, or 55). How many Legos did you have to start?</em></td>
</tr>
<tr>
<td><strong>Part-Part-Whole</strong></td>
<td><em>(Whole Unknown, PPW-WU)</em> <em>(Student's name) has (3, 8, 20, or 38) red beads and (5, 4, 30, or 49) blue beads. How many beads do you have?</em></td>
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<td></td>
<td><em>(Part Unknown, PPW-PU)</em> <em>(Student's name) has (8, 12, 50, or 87) beads. (3, 8, 20, or 38) are red the rest are blue. How many are blue?</em></td>
</tr>
<tr>
<td><strong>Compare</strong></td>
<td><em>(Difference Unknown, CDU)</em> *(Student's name) has (7, 70, or 78) matchbox cars. <em>(Friend's name) has (5, 50, or 57) matchbox cars. Who has more? How many more?</em></td>
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<td></td>
<td><em>(Compare Quantity Unknown, CQU)</em> *(Student's name) has (5, 50, or 57) matchbox cars. *(Friend's name) has (2, 20, or 21) more than <em>(Student's name).</em> How many matchbox cars does <em>(Friend's name)</em> have?</td>
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<td></td>
<td><em>(Referent Unknown, CRU)</em> *(Student's name) has (7, 70, or 78) matchbox cars. She has (2, 20, or 21) more matchbox cars than <em>(Friend's name).</em> How many matchbox cars does <em>(Friend's name)</em> have?</td>
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<tr>
<td><strong>Multiplication and Division</strong></td>
<td><em>(Multiplication, M)</em> <em>(a) There are 4 firefighters wearing their firefighting uniforms. How many boots are there?</em></td>
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<td></td>
<td><em>(b) There are (5 or 12) lizards in a pet store. How many lizard legs are there?</em></td>
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<td></td>
<td><em>(Measurement Division, MD)</em> <em>(If there are (10 or 50) suckers and each of <em>(Teacher's name)</em>'s friends gets (2 or 10) suckers, how many friends does <em>(Teacher's name)</em> have?</em></td>
</tr>
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<td><em>(Partitive Division, PD)</em> <em>(You have (12, 39, or 90) candy bars and you give them to 3 friends so they each get the same number. How many do they each get?</em></td>
</tr>
</tbody>
</table>

*Adapted from Carpenter et al. (2015)*

10 Illinois Mathematics Teacher
<table>
<thead>
<tr>
<th>Strategies that Student Uses</th>
<th>Not Yet</th>
<th>Sometimes</th>
<th>Usually</th>
<th>Teacher’s comments</th>
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<td>Skip Counting</td>
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<td>Counting strategies with number sets 20–100</td>
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<td>Counting strategies with number sets 0–20</td>
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<td>DERIVED FACTS</td>
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<td>NUMBER FACTS</td>
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<td>Needs Help</td>
<td>Sometimes</td>
<td>Usually</td>
<td>Teacher’s comments</td>
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<td><strong>Mathematical Practices for Student to Develop</strong></td>
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<tr>
<td>1) Makes sense of problems and perseveres in solving them by attempting a solution strategy.</td>
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<td>2) Reasons quantitatively and abstractly by making sense of quantities and the relationships among them such as realizing $5 + 7 = 5 + (5 + 2)$.</td>
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<td>3) Constructs viable arguments and critiques the reasoning of others such as having students know that when classmates’ explanations are unclear, students are to ask questions to make sense of the reasoning of others.</td>
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<td>4) Models with mathematics such as using pictures and numerical expressions to represent understanding of the problem.</td>
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<td>5) Uses appropriate tools strategically such as knowing when and how to use base-ten blocks.</td>
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<td>6) Attends to precision by communicating precisely using correct terms and by calculating accurately.</td>
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<td>7) Looks for and makes use of structure such as using the inverse, commutative, or distributive property appropriately.</td>
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<td>8) Looks for and expresses regularity in repeated reasoning, such as knowing that $3 \times 1 + 4 \times 1$ connects to $3 \times 10 + 4 \times 10$.</td>
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Appendix C

Strategies Students Could Use for Each of the Problem Types

Descriptions of strategies that go with each Problem Type are given below. These descriptions connect to Appendix B as listed on the Individual Assessment Sheets. For more details of the strategies refer to Carpenter et al. (1999) or Carpenter et al. (2015). For the strategies listed below, refer to Appendix A when examples are provided using specific numbers. This list is not all-inclusive; students may use other strategies than those listed.

Join Result Unknown (JRU)
- Join All by joining two sets of counters.
- Count On From First by using the first number given.
- Count On From Larger by starting with the larger of the two numbers and Counting On.
  - using knowledge of Number Facts and place value, e.g., grouping by tens.

Separate Result Unknown (SRU)
- Separate From by taking away objects from the initial set given.
- Count Down 2, 10, 20 or 35 by ones.
  - using knowledge of Number Facts and place value by Counting Down by groups of tens when given multi-digit numbers.

Join Change Unknown (JCU)
- Join To with counters having to make the initial set.
- Count On To with fingers or counters, not having to make the initial set.
  - using a strategy that does not connect to the structure of the problem which indicates flexibility of thinking, e.g., since the structure of the problem is addition, recognizing and using knowledge of the inverse relationship of subtraction.

Separate Change Unknown (SCU)
- Separate To with counters or fingers by starting with 9 counters and separating counters until 7 is reached, the answer being the counters in the set being separated.
- Count Down To with fingers or counters and mentally keeps track of the first number and the number counted down.
  - using a strategy that does not match the structure of the problem, thus indicating flexibility of thinking, e.g., since the structure of the problem is subtraction, recognizing the inverse relationship of addition indicates maturity of thinking: the student adds 2 to 7 to get 9 and knows the 2 represents the unknown.

Join Start Unknown (JSU)
- These problems can only be solved with understanding by students who recognize inverse relationships. Often students memorize these relationships rather than think about them. Solving this problem by thinking about relationships reflects maturity of thinking.
• The Direct Modelers and Counters can only use a Trial and Error strategy to solve this Problem Type. Thus they choose a number to start with and join counters or count on from the first number in the problem to determine if they get the final number in the problem. They use Trial and Error until a solution is reached.

Part-Part-Whole, Part Unknown (PPW-PU)

• Since this Problem Type does not involve action, it is more difficult to solve for the Beginning and Developing problem solver.
• A Mature problem solver could use a Joining To, Separating From, Counting On To, or Counting Down strategy.

Compare

• Matching by representing both sets with counters and counting the extra counters in the set that has more.
• Join All or Count On is used by Mature Problem Solvers.
• “There is not a commonly used strategy corresponding to the action or relationship described in the problem” for these Problem Types (Carpenter et al., 2015, p. 25).

Multiplication (M)

• Group by making 4 sets of 2.
• Skip Count by twos.
• using a Number Fact, e.g., \(4 \times 2 = 2 \times 4\) if the student understands the Commutative Property.

Measurement Division (MD)

• counting out ten suckers and putting them in groups of 2, counting 5 groups to discover there are five friends, Direct Modeling, Measurement strategy.
• double counting by putting 2 in a group and counting until 10 counters are used and then counting the groups and saying, “five,” Direct Modeling, Measurement strategy.
• Skip Count by 2’s until reaching 10, holding up a finger for each count, and realizing five fingers represent five friends.
• Knowing the Number Fact \(10 \div 2 = 5\) or \(10 \div 5 = 2\).

Partitive Division (PD)

• counting out 12 manipulatives and dealing counters to 3 groups until all counters are used and then counting how many are in each group to get the unknown 4; or by drawing a picture of 3 friends and giving one candy bar to each, then two, then three, then four as the student counts to 12 and then counts the number in each group to get 4, Direct Modeling, Partitive strategy.
• Using Trial and Error by picking a number such as 2 to skip-count, and saying “2, 4, 6” and realizing that won’t work, then trying “3, 6, 9” and then “4, 8, 12” and realizing that works so the unknown is “4.”
Appendix D
Assessment Flow Chart

Join (Change Unknown)

Solves using a Direct Modeling strategy

Join (Change Unknown) No counters

Solves using a Counting strategy

Separate (Result Unknown)

No Counters

Record student’s strategy

Have the student use counters to make sets of numbers between 1 and 20; record results

Join (Result Unknown)

Separate (Result Unknown)

Cannot Solve

Have the student use counters to make sets of numbers between 1 and 20; record results

Join (Result Unknown)

Separate (Result Unknown)

Cannot Solve

Record student’s strategy

Grouping (M)

Partitioning (MD)

Join (Change Unknown)

Record student’s strategy

Grouping (M)

Partitioning (MD)

Record student’s strategy

Partitioning (PD)

Part-Part-Whole (Part Unknown)

Record student’s strategy

Compare

Record student’s strategy

Separate (Start Unknown)

Record student’s strategy

Record student’s strategy

Continue with counters available

Cannot solve

Grouping (M)

Partitioning (MD)

Partitioning (PD)

Part-Part-Whole (Part Unknown)

Record student’s strategy

Partitioning (MD)

Record student’s strategy

Partitioning (PD)

Record student’s strategy

Record student’s strategy

Record student’s strategy