Abstract

Although sketching and building linear functions surfaces in many mathematics classes, taking direct measurements leading to function formulation is not often exercised. This paper—written in the form of a lesson accompanied by commentaries—is to fill in the gap. Guided by constructivist learning theory, the lesson places the students in the roles of scientists who integrate their mathematical theoretical knowledge with their skills of measuring to produce algebraic representations of straight lines. The unit does not involve complex equipment, except rulers that are readily available in any math classroom. Despite its simplicity, this lesson generates an engaging, discovery type learning environment that involves not only the students but also the teacher.

Keywords: mathematics, education, constructivism, slope

1. Introduction

Several studies (e.g., Stanton & Moore-Russo, 2012; Lobato & Siebert, 2002) have revealed that the concept of slope, despite its multiple applications, appears to students as an abstract mathematical entity. It is hypothesized that this difficulty might be rooted in presenting the slope as a ratio of dimensionless quantities, which diminishes its physical interpretation and weakens its relevance to students’ prior experiences. This paper presents a lesson that is designed to enrich the meaning of slope—and consequently the meaning of linear functions—by applying direct measurement.

The lesson is rooted in the constructivist theory of knowledge acquisition. Constructivists claim that learners construct their own knowledge based on received impulses (Von Glasersfeld, 1995). One of the opportunities to apply this learning theory is to place learners in realistic settings that resonate with their prior experiences and let them construct the knowledge.

The lesson presented here will reflect the constructivist approach. It can serve as a unit summarizing the process of building linear functions. It can be conducted in any math class provided students are familiar with the concepts of slope and the algebraic formulation of linear functions using slope-point or slope-intercept form.

2. Lesson Description

2.1. Lesson Outline

The commentary provided in the body of the manuscript is based on a lesson conducted with a group of precalculus students in a rural East Texas high school. During the lesson, the students—guided by the teacher—unfolded the physical meanings of all parameters and processes needed to express a given line in a symbolic algebraic form. A metric ruler was used to quantify the needed parameters, such as slope and intercepts. Measurement, which is an element of scientific inquiry, has received significant attention in the newly developed Common Core State Standards (Porter et al., 2011). Thus, including this element in the lesson enhanced not only the constructivist approach but also reflected new mathematics teaching recommendations. In addition, the students used graphing calculators to verify the derived mathematical models.

Since measuring is intertwined with the algebraic meanings of these processes in this lesson, it was hypothesized that a unified
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view of expressing a physical object—a line—algebraically would emerge in students’ minds as a result. It was also expected that, during this process, the mathematical tools used to express the slope of a line in mathematical form would become tangible and more meaningful to students.

The lesson made use of guided discovery as the method of instruction. Thus, the teacher was available to guide the students through the process of transferring a sketched line into its algebraic form.

2.2. Using Measurement to Connect with Students’ Prior Experiences

With the blackboard cleaned of any coordinate axes, the teacher opened the lesson by drawing a line on the board and positing the following question: how can one construct an algebraic equation of the line? The students were puzzled at first because no axes or grid lines were provided—i.e., they experienced a dissonance that prompted a discussion. The teacher directed students’ attention to a general structure of a linear function, $y - y_1 = m(x - x_1)$, and asked how to find the parameters that are needed to use the structure. Students suggested that selecting any two points from the line was necessary. The teacher selected two points and labeled them on the line (see figure 1) and then asked for the next step needed to find the line’s equation. The students were again in doubt because, even though the points were labeled, the coordinates were still not quantified, i.e., the line was placed in an environment without associated mathematical representations that the students were used to having provided for them.

The teacher asked what was needed to establish these points’ numerical coordinates. The students realized that a frame of reference called the “$x$-$y$ coordinate axes” was missing. The teacher then asked where exactly the axes should be drawn. The students suggested locating them near the line. The teacher drew, or asked a student to draw, the coordinate axes on the board in a standard vertical and horizontal fashion, as depicted in figure 2.

Figure 1: Line with two points identified.

Figure 2: Line with coordinate axes.
Since neither the scaling of the axes nor grid lines were provided, the students experienced yet another dissonance resulting in their inability to continue with the process of the line quantification. Some students suggested “making up a scale” on the $x$ and $y$ axes. The teacher asked if there was a more precise method of finding exact numerical magnitudes of the coordinates. The students had difficulty associating the coordinates with their measurable physical lengths. In order to make the transition, the teacher referred to a meter-stick and said that since the line was positioned with reference to the established $x$-$y$ axes, certain quantities such as lengths could be measured in order to quantify the inclination known as slope and the line’s exact location with reference to the axes. This was a pivotal element of the lesson and apparently “an eye opener” for some students. They realized that the commonly given coordinates represent, in fact, physical distances of points from an established reference, called the origin in mathematics.

A short discussion of what measuring system to use—initiated by the teacher—also took place. Students realized that whether they used centimeters or inches, the equation of the line, although expressed in different units, should resemble the same position and inclination as the line sketched on the board.

In order to fully experience this idea, the students were further guided through the process of deciding what segments to measure and how to measure the segments (see figures 3 and 4). The teacher asked some students to participate and measure the necessary lengths.

The students decided to quantify the vertical intercept first. Its magnitude as measured from the origin was 20 cm. They concluded that the next step was obtaining a numerical value for the slope, which was possible in two ways—either by measuring the coordinates of the points or by measuring the lengths of the run and rise between the points. Most of the students suggested measuring the run and rise between the points. The teacher sparked conversation by asking if measuring the length of the segment $AB$ would be significant in the process of slope quantification. Most students rejected this idea. Yet, as a right triangle was “formed,” this question generated a further inquiry about the differences between the concepts of slope and hypotenuse. Not all of the students understood that these two fundamental concepts were different. After a short discussion consensus was reached that the segment $AB$, when representing the slope, was a ratio, while, as a hypotenuse of a triangle, it represented a length.

2.3. Integrating the Measurements into Symbolic Line Representation

The inclination of the line was determined by the ratio of the vertical and horizontal distances between the points, i.e.,

$$\text{slope} = \frac{\text{vertical displacement}}{\text{horizontal displacement}} = \frac{\Delta y}{\Delta x}.$$
In the example, the slope took the value of \( \frac{43 \text{ cm}}{50 \text{ cm}} = 0.86 \). By applying the slope-intercept form, the students concluded that the equation of the line was \( y = 0.86 \frac{\text{cm}}{\text{cm}} x + 20 \text{ cm} \), or more simply, \( y = 0.86x + 20 \). The students were then asked to use a graphing calculator to verify that the graph on the board resembled the function equation. A discussion about selecting a calculator view window that would provide a full view of the line and also reflect the blackboard dimensions ensued. This discussion helped students to realize that the view window that would match the blackboard was determined by the horizontal and vertical distances of the established \( x-y \) axes from the edges of the blackboard. A comment supporting the use of negative values for the horizontal distance to the vertical axis on the left (\( X_{\text{min}} \)) and for the distance below the horizontal axis (\( Y_{\text{min}} \)) completed the discussion. The teacher concluded that the students’ term negative distance is not being used as in it is in physics, but instead a negative position is implied. This comment was especially of interest to students who were concurrently taking a physics course. It was also interesting to note that the position of the line was already determined with reference to the \( x-y \) axes, thus no further consideration of the line was needed to have it generated by the graphing technology. Typing the equation into the graphing calculator was sufficient. Students found it rewarding that the technology provided a line similar to that shown on the blackboard.

3. Interpreting the Slope

Just as the tangible tasks of measuring lengths can enhance slope quantification and help students to construct meaning, linking the tasks to applications and function monotonicity can expand student understanding of the slope’s significance.

3.1. Interpretation of the Units of Slope

It was important to note that in the process of algebraically expressing the line, the units of the lengths that constituted the slope were cancelled out, and therefore only the slope magnitude was used. This is the special case of dimensionless linear function representations, which are used widely in mathematics classrooms. Although the alternative of linking slope’s geometric meaning to its interpretation when the axes represent different quantities was not highlighted during this lesson, the following questions could inspire such an extension. For instance, could a speed of \( \frac{5 \text{ m}}{\text{s}} \) be perceived as ratio of lengths in which \( \frac{\text{m}}{\text{s}} \) is viewed in a manner similar to the geometric representation \([5 \text{ cm/m}] \frac{\text{m}}{\text{s}}\)? Would this representation lead the students to correct slope interpretations while simultaneously helping them retain the slope geometrical meaning? It seems that this idea is worth of further research.

3.2. Interpretation of the Slope’s Sign

The lesson also provided opportunities for highlighting the association between the sign of the slope and function increase or decrease. From this point, we use the general slope definition \( \frac{\Delta y}{\Delta x} \), rather than the less formal \( \frac{\text{rise}}{\text{run}} \). In the analyzed example, the value was \( \frac{\Delta y}{\Delta x} = 0.86 \). Solving for \( \Delta y \), we have \( \Delta y = 0.86 \Delta x \). This statement indicates that for every increase in the horizontal distance by, for instance, 1 cm, the vertical height of the line increases by 0.86 cm \((\Delta y = 0.86(1 \text{ cm}) = 0.86 \text{ cm})\), and this positive slope causes an increase of the function values. Respectively, if one obtains the slope to be, for example, \(-2\), then this will be interpreted to mean that for every increase of \( x \) by one unit, the \( y \) coordinate of the function decreases by 2 units according to \( \Delta y = -2 \Delta x \). The slope appears as an indicator of how the function’s \( y \)-coordinates change. Consequently, this interpretation will enhance the calculus interpretation of the derivative (often called the slope function by students), which says that if the derivative of \( f(x) \) is negative on a given interval (and therefore the instantaneous slopes are negative), the values of the function will decrease on the interval and vice versa (e.g., see \cite{Stewart2007}).
3.3. Example Combining Both Interpretations

When mathematizing the slope, the students realized that the formula \( \frac{\text{rise}}{\text{run}} \) constitutes a ratio of two lengths—the horizontal separation between the selected points and their vertical separation. Seeing the units of lengths in the slope quantification helped initiate students’ transition to thinking in other contexts. The teacher posed the following question: suppose that the slope of a line representing water temperature measured in degrees Fahrenheit versus time over some time interval is 9. Then suppose that the water temperature was labeled on the vertical axis and the time of heating the water, in minutes, was labeled on the horizontal axis. What is the geometric and scientific interpretation of this slope? The students realized that a positive slope indicated a ratio of a positive rise over positive run (it was suggested that run was always considered positive). In a scientific interpretation, it meant that the water temperature was increasing by 9°F for every minute of heating.

3.4. Suggestions for Independent Student Practice

Providing students with other contexts in which the slope can be calculated will broaden the idea and enhance its relevance. The following are some suggested extensions.

1. Use simulations (see, for example, the Balancing Act simulation on the PhET website, http://phet.colorado.edu/en/simulation/balancing-act) to copy and paste a few snapshots and have students calculate the slopes and define the corresponding function equations.

2. Provide students with various triangles cut out from a piece of cardboard. Have the students trace the triangles, determine the coordinates, and then construct the mathematical equations of the selected inclinations.

3. Supply students with fixed linear graphs representing various quantities and ask the students to determine function parameters from them.

4. Reflections

This lesson arose from the author’s search to contextualize mathematical ideas and create learning environments according to the contemporary constructivist learning theory. As a novice trying a simple approach, the lesson generated several reflections. Despite being conducted in precalculus classes—where linear functions play a rather marginal role—it turned out to be an engaging and meaningful learning experience for the students and for the teacher. As mathematics concepts are usually presented to students deductively by providing fixed formulas and practicing their applications, this lesson brought in elements of investigation that will be meaningful in the process of function formulation. The students were surprised, for example that \( x-y \) coordinates can be physically measured. Their experience deepened when they observed—on their graphing calculators—a line that resembled the one originally sketched on the board by the teacher. This gave them an evident proof that abstract math concepts are tangible and applicable in reality. By using measurement, the students seemed to find meaning and became encouraged to construct the function on their own.

The question remains: what should be the appropriate grade level to introduce such lesson? It is hypothesized that the most appropriate grade level is when students are introduced to the idea of linear functions for the first time. It is further hypothesized that such a lesson will provide students with a reference for sketching linear functions in their further math courses. The author would like to encourage readers to trying to conduct this lesson and share their reflections.

References


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