Chips, Chopsticks, and Polygonal Numbers to Foster Algebraic Thinking

Jeong Oak Yun – Seoul Global High School, lena.yun@gmail.com
Alfinio Flores – University of Delaware, alfinio@math.udel.edu

In these activities, students represent triangular, square, pentagonal and hexagonal numbers with chips. Chopsticks are used to break polygonal numbers into components. By representing each part with an algebraic expression, students can find an algebraic representation for the total, and establish relations between different algebraic expressions. In this article we focus on breaking pentagonal and hexagonal numbers into triangular numbers. Of course other decompositions of polygonal numbers are possible and are hinted at the end of the article. The goals of these activities are to help students

- find number patterns that have geometrical structure;
- develop their own strategies to count numbers that form a pattern;
- compare their strategies and learn from each other; and
- experience the beauty of mathematics.

The activities were conducted with five groups in a public magnet school in Korea. The students are among the top 20% of 11th grade students and have special talent for English and preference for social studies, but they are not especially gifted in mathematics. For the first activity the teacher illustrated how to represent the triangular numbers algebraically after using chopsticks to partition geometrically arranged chips and finding the numerical representation of the 2nd, 3rd, and 4th triangular numbers. Then the teacher explained how to form pentagonal numbers with chips, she asked her students to find their own strategies for using chopsticks to partition them and finding numerical and algebraic representations of the pentagonal numbers. Then she asked students to do the same for central hexagonal numbers. The teacher prepared a PowerPoint presentation and asked students to draw imaginary chopsticks on the board where the slides were projected (Figure 1).

Students first wrote the numeric representation of 2nd, 3rd, and in some cases 4th polygonal numbers, then wrote an algebraic representation. Students enjoyed finding many different strategies for just one question and even became competitive. In one class students found eight different ways to represent pentagonal numbers. In another class students were eager to share their strategies even after the bell had rung. Some of the students who did not show much interest in the beginning, after watching their peers present their novel ideas, became interested and tried to come up with their
own ideas. In some cases students were amazed by the different strategies of their peers. Although students were not particularly fond of the chopsticks and wanted to use tools that could be bent like wire, thinking about where to put the chopsticks seemed to help them to concentrate on how to partition the array of chips. Using tools to partition the arrays was useful in finding patterns.

Activity 1: Triangular Numbers

1. The sums of consecutive whole numbers like 1, 1 + 2, 1 + 2 + 3, ..., can be represented as triangular shapes (Figure 2). Represent the first four triangular numbers with poker chips.

![Figure 2. The first four triangular numbers](image)

![Figure 3. The $n^{th}$ triangular number](image)

2. Make two copies of each triangular number with chips of two colors, and arrange them together to form rectangular arrays (see Figure 4). Describe the number in each rectangle as the product of the number of chips in each row by the number of rows. Express the number of chips for each triangular number as half the chips in the corresponding array. Generalize for the $n^{th}$ triangular number.

$T_1 = \frac{1 \times 2}{2}$

$T_2 = 1 + 2 = \frac{2 \times 3}{2}$

$T_3 = 1 + 2 + 3 = \frac{3 \times 4}{2}$

$T_4 = 1 + 2 + 3 + 4 = \frac{4 \times 5}{2}$

$T_n = 1 + 2 + 3 + \cdots + n = \frac{n \times (n + 1)}{2}$

Activity 2: Square Numbers

The arrays in figure 5 are called square numbers. The number of chips in each row is equal to the number of rows. So, if there are $n$ rows, the total amount of chips will be $n \times n = n^2$. Use chips to build the first four square numbers.

![Figure 5. Square numbers](image)

Find triangular numbers within the square numbers and use a chopstick to separate them (Figure 6).

![Figure 6. Triangular numbers and square numbers](image)

Express each square number algebraically as the sum of two terms that represent triangular numbers. For example,

$S_3 = 3 \times 3 = \frac{2 \times 3 + 3 \times 4}{2}$

$S_4 = 4 \times 4 = \frac{3 \times 4 + 4 \times 5}{2}$

Generalize to the $n^{th}$ square number. Verify that the algebraic expressions on both sides of the following identity are indeed
equivalent. Expand and simplify the right side.
\[ n \times n = \frac{(n-1) \times n + n \times (n+1)}{2} \]

Activity 3: Pentagonal Numbers

The arrays in figure 7 represent pentagonal numbers. Each new pentagonal number is formed by adding a new layer consisting of three sides at the bottom, thus extending the two sides that meet at the upper vertex. This upper vertex has thus a different role than the other vertices in this kind of pentagonal array.

Figure 7. Pentagonal numbers

Pentagonal numbers can be decomposed into triangular numbers. Figure 8 shows one kind of decomposition.

Figure 8. Pentagonal numbers and triangular numbers.

Express each of the pentagonal number as a sum of triangular numbers. For example,
\[ P_2 = 1 + 1 + 3 = 2 \times \frac{1 \times 2}{2} + 2 \times \frac{2 \times 3}{2} \]
\[ P_3 = 3 + 3 + 6 = 2 \times \frac{2 \times 3}{2} + 2 \times \frac{3 \times 4}{2} \]

Figure 8. The \( n \)th pentagonal number

Express the \( n \)th pentagonal number as the sum of the \((n-1)\)th triangular number and the \( n \)th square number. Verify that this algebraic expression is equivalent to the one obtained above.

Students found several partitions and the corresponding algebraic expressions. Verify that indeed the total number of chips is the same. For the partition in Figure 7 students found this expression
\[ 1 + (3 \times 2 - 2) + (3 \times 3 - 2) + (3 \times 4 - 2) + \cdots = \sum_{k=1}^{n} (3k - 2) \]

Figure 12. \[ \frac{n(n+1)}{2} \times 2 - 1 + \frac{(n-2)(n-1)}{2} \]
Students used “wire” for the following partition.

Activity 4: Hexagonal Numbers
The arrays in Figure 15 are called the central hexagonal numbers.

Use chopsticks to break each hexagonal number into triangular numbers. Several solutions are possible. Figure 16 shows one.

Express the hexagonal numbers in terms of the triangular numbers.

\[ H_3 = 1 + 6 \times \frac{2 \times 3}{2} \]
\[ H_4 = 1 + 6 \times \frac{3 \times 4}{2} \]

Write an expression for the number of chips in the fifth central hexagonal number.

Generalize to the \( n^{th} \) central hexagonal numbers, \( H_n = 1 + 6 \times \frac{(n-1) \times n}{2} \).

Use the partition of \( H_4 \) illustrated in Figure 17 to suggest another algebraic expression. Express first this particular hexagonal number as the sum of particular triangular numbers and then generalize. Verify that the new general expression is equivalent to the one obtained above.

Below are other partitions found by students and the corresponding algebraic expressions.
Figure 21 illustrates the fourth central hexagonal number as the sum of hexagonal shells. Students found an expression for the \( n \)th central hexagonal number as the sum of the shells \( 1+(6 \times 1)+(6 \times 2)+(6 \times 3)+\cdots=1+6 \times \sum_{k=1}^{n-1} k \).

Verify that this expression is equivalent to the expressions obtained above.

Figure 21. The fourth central hexagonal number

Final Comments

Polygonal numbers are one kind of geometrical representation of numbers and relations among numbers. Students can use such geometrical representations as a means to explore algebraic ideas. With the help of these representations students can think about the relations among the numbers, express them using their own words, and represent them with letters. These activities can stimulate students to try to find various ways of solving a problem and appreciate the joy of finding various solutions. The activities also foster them to think how to find patterns, to express the patterns in numerical forms, and to generalize them into algebraic forms. A teacher can use geometrical representations to help students as they learn to use algebra to generalize and justify (Flores 2002). Polygonal numbers and other geometrical representations can provide a more concrete step towards the more abstract use of letters as variables or generalized numbers, which for beginners may be a little complicated.

References

