What if Not? : Connecting Probabilities Involving the Sum of Dice

William Beggs, Glenbard North High School, retired, williambeggs@gmail.com
Dane Camp, New Trier High School, campd@newtrier.k12.il.us

In the study of probability at the high school level, most students will solve problems involving the sum of two dice. A typical problem might be to find the probability that the sum is eleven when two dice are tossed. Teachers will often vary this problem by asking students to find the probability that the sum is less than eleven, or that the sum is seven or eleven. A creative way to generate more challenging problems is to use the “What if Not?” technique.

Stephen Brown and Marion Walter, at Harvard University, in the late nineteen-sixties developed this technique. The “What if Not?” technique begins by listing attributes of a given problem, theorem, or puzzle. For example consider the opening problem:

What is the probability when rolling two six-faced dice that the sum is eleven?

Six attributes for this problem are listed below.

1. Two dice are being tossed.
2. The sum is eleven.
3. The dice have six faces.
4. The sum is being calculated.
5. The dice are fair.
6. The number system is base ten.

The second stage of this technique is to negate these attributes one at a time and then, based on each negation, attempt to generate a new problem. For example suppose that the first attribute is negated. New problems would deal with sums of eleven where more than two dice are being tossed. A possible way to pose a new problem is as follows:

In a math classroom there are eleven six-sided dice sitting on top of a desk. A student enters the classroom and selects some of these dice from the desk. The student rolls the dice and obtains a sum. If the student wants the greatest probability of rolling a sum of eleven, then how many of these dice should the student select?

I have chosen to include the case of tossing two dice with the other possible cases. It is impossible with a single die to attain a value of eleven, and with more than eleven dice the sum will always be larger than eleven. The remainder of this article will tell the story that unfolded in an attempt to find a solution to this problem.

When I originally solved this problem I started by looking at ordered n-tuples and used permutations to help calculate these probabilities. For example with the case of three dice, I first found six triples where the sum was eleven. The six triples were (1, 4, 6), (1, 5, 5), (2, 3, 6), (2, 4, 5), (3, 3, 5), and (3, 4, 4). I then found all ordered triples that could be generated from these original six. The chart on the next page shows the 27 possible solutions for which the sum of three dice is eleven.
The number of ways that the three dice can fall is $6^3$ or 216. The probability of tossing three dice and having a sum of 11 is $\frac{27}{216}$ or $\frac{1}{8}$. It might be noted that for the three triples $(1, 5, 5)$, $(3, 3, 5)$, and $(3, 4, 4)$, the repeated digits are not distinguishable and therefore, there are fewer triples that can be determined from these.

A similar approach was used to find the probabilities for 4-tuples, 5-tuples, and continuing all the ways up to 11-tuples. The following table shows the resulting probabilities.

<table>
<thead>
<tr>
<th>Original Triple</th>
<th>New Triples</th>
<th>Counting Principle</th>
<th>Number of Dice</th>
<th>Probability (Sum=11)</th>
<th>3-Significant Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 4, 6)$</td>
<td>$(1, 6, 4)$, $(4, 1, 6)$, $(6, 1, 4)$, $(6, 4, 1)$</td>
<td>$3! = 6$</td>
<td>2</td>
<td>$\frac{2}{36} = 0.0556$</td>
<td>0.0556</td>
</tr>
<tr>
<td>$(1, 5, 5)$</td>
<td>$(5, 1, 5)$, $(5, 5, 1)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>3</td>
<td>$\frac{27}{216} = 0.125$</td>
<td>0.125</td>
</tr>
<tr>
<td>$(2, 3, 6)$</td>
<td>$(2, 6, 3)$, $(3, 2, 6)$, $(3, 6, 2)$, $(6, 2, 3)$, $(6, 3, 2)$</td>
<td>$3! = 6$</td>
<td>4</td>
<td>$\frac{104}{1296} = 0.0802$</td>
<td>0.0802</td>
</tr>
<tr>
<td>$(2, 4, 5)$</td>
<td>$(2, 5, 4)$, $(4, 2, 5)$, $(4, 5, 2)$, $(5, 2, 4)$, $(5, 4, 2)$</td>
<td>$3! = 6$</td>
<td>5</td>
<td>$\frac{205}{7776} = 0.0264$</td>
<td>0.0264</td>
</tr>
<tr>
<td>$(3, 3, 5)$</td>
<td>$(3, 5, 3)$, $(5, 3, 3)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>6</td>
<td>$\frac{2}{36} = 0.00540$</td>
<td>0.00540</td>
</tr>
<tr>
<td>$(3, 4, 4)$</td>
<td>$(4, 3, 4)$, $(4, 4, 3)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>7</td>
<td>$\frac{210}{6^7} = 7.50 \times 10^{-4}$</td>
<td>$7.50 \times 10^{-4}$</td>
</tr>
<tr>
<td>$(3, 5, 3)$</td>
<td>$(5, 3, 3)$, $(3, 3, 5)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>8</td>
<td>$\frac{120}{6^8} = 7.14 \times 10^{-5}$</td>
<td>$7.14 \times 10^{-5}$</td>
</tr>
<tr>
<td>$(4, 3, 3)$</td>
<td>$(3, 3, 4)$, $(3, 4, 3)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>9</td>
<td>$\frac{45}{6^9} = 4.47 \times 10^{-6}$</td>
<td>$4.47 \times 10^{-6}$</td>
</tr>
<tr>
<td>$(4, 3, 4)$</td>
<td>$(4, 3, 4)$, $(4, 4, 3)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>10</td>
<td>$\frac{10}{6^{10}} = 1.65 \times 10^{-7}$</td>
<td>$1.65 \times 10^{-7}$</td>
</tr>
<tr>
<td>$(4, 4, 3)$</td>
<td>$(4, 3, 4)$, $(4, 4, 3)$</td>
<td>$\frac{3!}{2!} = 3$</td>
<td>11</td>
<td>$\frac{1}{6^{11}} = 2.76 \times 10^{-9}$</td>
<td>$2.76 \times 10^{-9}$</td>
</tr>
</tbody>
</table>

From the probabilities shown in the table a solution to the problem can be found. The greatest probability is when the student selects three dice. The second best option is for the student to select four dice. The third highest probability will result when a student selects just two dice.

My second approach to solving this problem involved the use of combination numbers. I first designed a table like the one shown on the next page. The values in the table represent the number of ways that a sum can be obtained for a given number of dice. For example, in tossing 3 dice there are 10 triples whose sum is 6. 10 is the underlined value in the table on the next page.
I noticed that the values shown in each column represented a diagonal in Pascal’s Triangle. As shown in the diagram below, each number in Pascal’s Triangle is a combination number.

```
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
```

From the diagram, \( _2C_2 = 1, _3C_2 = 3, _4C_2 = 6, \) and \( _5C_2 = 10 \). If the pattern continues then the sum of 11 on 3 dice should be \( _{10}C_2 = 45 \). However, the answer to this problem as shown earlier was 27. Why did the pattern fail? The reason is that in counting the triples using this combination approach, the four triples \((7,1,3), (7,2,2), (8,2,1)\) and \((9,1,1)\) are included. Also all permutations of these four triples are included in the count of 45. There are eighteen triples that are counted that either contain a 7, 8, or 9. When rolling dice the largest integer value is 6. If these 18 triples are subtracted, then the result is 27 as was shown earlier.

I saw the following pattern for this problem:

\[
\begin{align*}
_2C_2 - 18 &= 27 \\
_3C_2 - 3(6) &= 27 \\
_4C_2 - 3(_4C_2) &= 27
\end{align*}
\]

In investigating for other cases, I noticed the following pattern. The results are summarized in the table below. Note that \( _sC_r = 0 \) for \( n < r \).

<table>
<thead>
<tr>
<th>Number of Dice</th>
<th>Probability (Sum=11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( _{10}C_1 - 2(_4C_1) )</td>
</tr>
<tr>
<td>3</td>
<td>( _{10}C_2 - 3(_4C_2) )</td>
</tr>
<tr>
<td>4</td>
<td>( _{10}C_3 - 4(_4C_3) )</td>
</tr>
<tr>
<td>...</td>
<td>: ( : )</td>
</tr>
<tr>
<td>11</td>
<td>( <em>{10}C</em>{10} - 11(<em>4C</em>{10}) )</td>
</tr>
</tbody>
</table>

Using the table I was able to generalize and write the following:

If \( n \) is the number of dice \((2 – 11)\) and the sum is eleven then the desired probabilities can be found using the following equation:

\[
\text{Probability (sum=11)} = \frac{_{10}C_{n-1} - n(_4C_{n-1})}{6^n}
\]

From this generalization I then realized that the list capability of the TI-84 calculator could be used to calculate and display these ten probabilities. The steps necessary to enter this into the TI-84 are outlined on the next page.
In List 1 (L₁) enter the values (2, 3, 4,..., 11)
Set: \( L₂ = \binom{10}{L₁+1} \), \( L₃ = L₁ \times \binom{4}{L₁-1} \)
\( L₄ = L₂ - L₃ \) \( L₅ = L₄ / 6^{L₁} \)

The values that are stored in L₅ are the set of ten probabilities for this problem. Below is a list of the first seven values that are stored in the lists (L₁ – L₅).

<table>
<thead>
<tr>
<th></th>
<th>L₁</th>
<th>L₂</th>
<th>L₃</th>
<th>L₄</th>
<th>L₅</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
<td>10</td>
<td>5</td>
<td>2</td>
<td>.05556</td>
</tr>
<tr>
<td>2</td>
<td>45</td>
<td>18</td>
<td>10</td>
<td>27</td>
<td>.125</td>
</tr>
<tr>
<td>3</td>
<td>120</td>
<td>18</td>
<td>104</td>
<td>104</td>
<td>.00325</td>
</tr>
<tr>
<td>4</td>
<td>240</td>
<td>5</td>
<td>205</td>
<td>205</td>
<td>.02886</td>
</tr>
<tr>
<td>5</td>
<td>540</td>
<td>0</td>
<td>252</td>
<td>252</td>
<td>.0054</td>
</tr>
<tr>
<td>6</td>
<td>120</td>
<td>0</td>
<td>120</td>
<td>120</td>
<td>.015</td>
</tr>
<tr>
<td>7</td>
<td>120</td>
<td>0</td>
<td>120</td>
<td>120</td>
<td>.015</td>
</tr>
</tbody>
</table>

The scatter plot is shown below.

To display a scatter plot of these ten probabilities, go to Stat plot and select the following on the Plot1 window.

Probability (sum=x) = \( \frac{x-1}{\binom{n-1}{x}} \times \frac{1}{6^n} \)

Unfortunately, this formula did not work in general. For example consider the case of tossing 3 dice and attempting to obtain a sum of 17. The formula yields the following:

\[
\text{Probability (sum=17)} = \frac{16 \times \binom{2}{3} - 3 \times \binom{10}{2}}{6^3} = \frac{120 - 135}{216} = \frac{-15}{216}
\]

Since the probability of an event cannot be a negative value, it is obvious that this formula does not produce the correct result. There are only three triples (5, 6, 6), (6, 5, 6), and (6, 6, 5) that have the sum of seventeen. The probability is 3/216 or 1/72.

In finishing with this problem, I wanted to see if I could write a probability distribution for the sum of two dice, three dice, four dice, and five dice. In writing this distribution I used the fact that these probability distributions are symmetrical about the mean. For a single die the mean is 3.5, and for n dice the mean is 3.5n. I also noticed that for n dice the first 6 probabilities ranging from n to n+5 can always be found using the formula

\[
P(\text{sum}=x) = \frac{x-1}{\binom{n-1}{x}} \times \frac{1}{6^n}
\]

Looking back at (Table 3) this means that the combination expressions shown in the row where the sum is x only work for the first six values in each column. For three dice this means that only sums of 3, 4, 5, 6, 7, and 8 can be used with this formula.
The probability distribution functions are shown below and it should always be assumed that \( x \) is an integer value.

**Two Dice**

\[
\begin{align*}
P(\text{sum} = x) &= \begin{cases} 
\binom{x-1}{C_1} & 2 \leq x \leq 7 \\
\binom{14-x}{C_1} & 8 \leq x \leq 12
\end{cases}
\end{align*}
\]

**Three Dice**

\[
\begin{align*}
P(\text{sum} = x) &= \begin{cases} 
\binom{x-1}{C_2} & 3 \leq x \leq 8 \\
\binom{x-2}{C_2} - 3\binom{x-7}{C_2} & 9 \leq x \leq 10 \\
\binom{21-x}{C_2} & 11 \leq x \leq 18
\end{cases}
\end{align*}
\]

**Four Dice**

\[
\begin{align*}
P(\text{sum} = x) &= \begin{cases} 
\binom{x-1}{C_3} & 4 \leq x \leq 9 \\
\binom{x-2}{C_3} - 4\binom{x-7}{C_3} & 10 \leq x \leq 14 \\
\binom{28-x}{C_3} & 15 \leq x \leq 24
\end{cases}
\end{align*}
\]

**Five Dice**

\[
\begin{align*}
P(\text{sum} = x) &= \begin{cases} 
\binom{x-1}{C_4} & 5 \leq x \leq 10 \\
\binom{x-2}{C_4} - 5\binom{x-7}{C_4} & 11 \leq x \leq 16 \\
\frac{780}{6^5} & x = 17 \\
\binom{35-x}{C_4} & 18 \leq x \leq 30
\end{cases}
\end{align*}
\]

The big surprise was for the case of 5 dice and a sum of 17. Using the formula \( P(\text{sum}=17) = \binom{x-1}{C_4} - 5\binom{x-7}{C_4} \), the result is \( 770 / 6^5 \). The actual answer is \( 780 / 6^5 \). At this point I wanted to be sure my calculations were correct. I sent a copy of my calculations together with a manuscript to Dane Camp.

The following is what Dane discovered while verifying my calculations.

When I got the draft of the manuscript in the mail, I was excited. Bill had introduced me to the “What If Not?” technique years before and I was looking forward to seeing another interesting example. Though I enjoyed reading as the application unfolded, my heart sank when I finished and looked at the pages attached, which included detailed computations enumerating each possibility. I am not a great computational proofreader and didn’t look forward to hours of plug and chug. Finally, while attempting to check the work on these sheets, I stumbled across what I thought was a generalization. Had I read this manuscript at some other time, I doubt if I would have noticed it but serendipity intervened. We had just recently finished doing combinatorial analysis in my honors precalculus class and I still had “ball and urn” problems and the principle of inclusion/exclusion running through my head. As I progressed through the sheets, I gradually refined a formula using these two principles.

Using Bill’s narrative as inspiration, I had a leg up on hunting for a generalization. I immediately thought of a ball and urn problem (I think the official parlance is partition, but I’m not sure.) So the question was: *How many ways can I distribute \( s \) balls (dots) into \( d \) urns (upturned faces on the dice) where each urn has at least one ball and at most 6 balls?*

For example, consider the case of rolling a sum of 17 using 5 dice. Each die must have at least one dot showing, so there are 17 – 5 dots left to distribute to the 5 dice.
Thus we can think of this as a “word” problem where we are looking for the number of ways to rearrange $17-5$ 0’s and 5 – 1 separators. One such arrangement would be 00000/000000/00/000/0 – representing 4, 6, 3, 3, and 1 (the bold zeros are “glued” to insure that there is at least one dot showing on each die). So far, we get the total number of ways to get the sum to be

\[
\binom{17-5}{(17-5)} + \binom{5-1}{5-1} = \binom{16}{12} = 1820. 
\]

So, in the general case we would glue one dot onto each one of the $d$ faces showing. For a sum of $s$, there are $s-d$ dots remaining to distribute. I think of this as what some call a “MISSISSIPPI” problem with $s-d$ 0’s and $d-1$ /’s. Thus we can think of this as a “word” representing 4, 6, 3, 3, and 1 (the bold zeros are “glued” to insure that there is at least one dot remaining to distribute to the (still) $d-1$ /’s.

In other words we must remove

\[
\binom{s-d}{(s-d)} + \binom{d-1}{d-1} = \binom{s-6-1}{s-d-6}
\]

from the previous total.

Here is where the principle of inclusion/exclusion kicks into the problem. For some instances there are cases that are double counted. In these scenarios, for example, it is possible to have more than six dots on one face and more than six on another. We have deducted this twice. This explains the discrepancy that Bill discovered while considering 5 dice summing to 17, the arrangement 0000000/000000/0/0/0 was actually deducted twice, once for violating having over six dots on the first die and once for violating having more than 6 dots on the second die. Since it was only supposed to be deducted once, we have to add it back one time. This error is repeated for every pairing of the five dice, so we must add back to the sum an amount equivalent to

\[
\binom{5}{2} \binom{(17-5-2\cdot6)+(5-1)}{17-5-2\cdot6} = \binom{5}{2} \binom{4}{0} = 10.
\]

Thus the grand total is $1820 - 1050 + 10 = 780$--just as Bill had noted when he was checking his result. In general, we must add back all of the possible pairings of $d$ dice taken two at a time where two of them have more than 6:

\[
\binom{d}{2} \binom{(s-d-2\cdot6)+d-1}{s-d-2\cdot6} = \binom{s-2\cdot6-1}{s-d-2\cdot6}.
\]

Naturally, when the sum is large enough, we must also consider triple overlap, quadruple overlap, etc. Using the principle of inclusion/exclusion, we know that the terms alternate between being added and subtracted. Since there are $d$ dice, we only need consider the sum up to $d$. (Luckily, the combinations come out to be zero when the overlap is empty.) Putting all of this together we get,
\[ \sum_{t=0}^{d}(\frac{d}{t})(\frac{s-6t-1}{s-d-6t}) \]

the formula that I used to check out all of Bill’s numbers.

But, even after I sent the manuscript back to him, this rich problem haunted me. Eventually, I noticed two other refinements. First, instead of using \( d \) as the upper bound on the summation parameter, it would be more efficient to employ the floor function \( \lfloor \frac{s-d}{6} \rfloor \). So, for example, a sum of 17 on 5 dice means that the maximum value on the summation is \( t = 2 \) not \( 5 \) (as we noted before) since any overlap beyond the double overlap is empty. Also, in the spirit of the “What If Not?” technique, it struck me that there is nothing sacred about cubical dice. If we stick to platonic solids, we just need to replace any 6’s with the appropriate number of faces. So for a tetrahedron, we could simply replace the 6’s with 4’s. In general, if we replace 6 with \( f \), the number of faces of the dice, we get a wonderfully compact

formula:

\[ \sum_{t=0}^{\left\lfloor \frac{s-d}{f} \right\rfloor}(\frac{d}{t})(\frac{s-f^t-1}{s-d-f^t}) \]

I can’t remember the last time I had so much fun tinkering with a single problem!

We now return to Bill’s account of the problem.

Using the generalization that Dane discovered, I was able to write the program DICESUM for the TI-84 calculator. This program will allow the user to enter the number of 6-faced dice and the sum that is desired. The program will then calculate the corresponding probability. This program can be easily modified so that the user could enter the number of desired faces on each die. The code for the program DICESUM is shown next.

```
PROGRAM : DICESUM
:ClrHome
:Disp "ENTER THE NUMBER"
:Disp "OF DICE"
:Input D
:Lbl 1
:Disp "ENTER THE SUM"
:Input S
:If (S<D) or (S>6*D)
:Then
:Goto 1
:Else
:0->W
:0->T
:Lbl 2
:S-(6*T)-1->A
:S-D-(6*T)->B
:(D \text{n}_C_T) \times (A \text{n}_C_B) \times (-1)^T->K
:W + K -> W
:T+1 -> T
:If T \leq \text{int}((S-D)/6)
:Then
:Goto 2
:Else
:6^D-> M
:W/M-> X
:Disp "SUCCESS NUM"
:Disp W
:Disp "TOTAL NUM"
:Disp M
:Disp "PROB"
:Disp X
:End
```

An example execution on the program is shown below. If the user attempts to enter a sum that is impossible to attain, the program will prompt the user with “ENTER THE SUM”. For very large input values of course there is the possibility of overflow.

```
ENTER THE NUMBER OF DICE 25
ENTER THE SUM 17
SUCCESS NUM 780
TOTAL NUM 7776
PROB .100380642
```

```
Done
```
As this story draws to a close there are a couple of points that I would like to make. The calculations needed to explore this problem did not unfold in a single day. I worked on this problem on and off for several months. A problem of this type can take a great deal of time to investigate and refine. Students should be aware that every math problem does not need to be solved in a short period of time.

I want to emphasize how teamwork was the recipe that eventually led to the discovery of the generalization. I would not have found the generalization on my own. I doubt that Dane would have investigated a problem of this type without seeing my initial calculations. There were some exciting moments as we traded emails regarding the construction of this manuscript.

There are still several unanswered questions. The first three of the six attributes were discussed during the course of this article. However, there are three that remain as challenges to the reader. For the computer programmer there are several possibilities for constructing programs that involve probabilities related to the sum of dice. To statistics teachers there are topics and notation that can be incorporated into lessons and shared with students.

In closing I give credit to the “What if not?” technique for helping this story to unfold. This technique provides an excellent framework for generating problems and promoting mathematical investigations. For this reason this strategy must be shared with students at all grade levels. I believe that many significant discoveries can be made by students who are exposed to the “What if Not?” technique.

Your presence is requested at the 59th Annual Meeting and Conference of the Illinois Council of Teachers of Mathematics, “Connections in the Math Landscape”. It will be held October 16-18, 2008 in Peoria, Illinois. We hope to see you there.