Developing a Mathematician's Mind by Requiring Reasoning and Pattern Finding

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Mathematics, as the study of patterns and the study of numbers, shapes and symbols, provides a way to look at the elegance and beauty of our world. Richard Feynman, a well known American physicist, stated, "To those who do not know Mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty of nature..." (Feynman, 1994). This article provides examples of students role-playing as mathematicians in problem-solving settings. Through these tasks children gain insight into the elegance and beauty of our world while developing mathematical reasoning and pattern-finding abilities. The atmosphere of students doing a task, or solving a "big" problem, has a sense of excitement in which students develop higher-level thinking and communication skills. (Bright, 1996; Chancellor, 1991; Kamii, 1993; Stanic, 1990).

In our district, students who show academic promise have an opportunity to work in small groups as part of an enrichment program. While the examples described below were done as part of an enrichment program, they are adaptable to other classroom settings. Teaching students to reason, problem-solve, and communicate in the manner of a mathematician or engineer is for every student! Mathematics enrichment lessons fall into three main categories: students doing an extension of a math lesson, often connected to story problems, students doing an interdisciplinary task which involves concepts related to the classroom curriculum, and students trying to answer an "imponderable problem."

Story Problems

Story problems are an essential part of the extensions to the classroom curriculum. The story problem can pose a scenario where students apply prior learning and practice mathematical thinking via the questioning process. The four phases initially suggested by the well known mathematician George Polya are used as a basis. These are enhanced with the following 4 questions (numbered below) developed by Art Hyde (Hyde, 1991).

Polya's Steps with Hyde's Questions

- Understanding the Problem.
  1. What do you know for sure?
  2. What do you want to find out?
  3. Are there any special rules or conditions?
- Devising a plan of attack
- Carrying it out
- Reviewing / Looking Back
  4. Is there another way to do this?

This process is modeled by our enrichment aides who work with the students. As students internalize the process, there is less and less modeling. In kindergarten and first grade, many of these problems are done directly with the enrichment aide. Since many
of these students’ thinking skills go beyond their reading skills, the assistance of a fluent reader, is also a factor in the type of groupings we plan. In second and third grade, students will usually be in pairs to solve problems and come together to discuss their answers.

In looking at “What do you know for sure?” questions of vocabulary can come out as well as identifying layers of meaning for various facts. In answering the question, "What are you trying to find out?" students can learn how to analyze patterns and identify relationships. Several kinds of practice are used to provide students with strategies to formulate questions. Alongside more traditional math story problems, a series of stories in the *Real Math Thinking Story Book* series by Stephen Willoughby and Carl Bereiter (1985) is used to teach reasoning skills. In these stories, students are to anticipate the questions a designated character will ask. Children develop their ability to reason through the text situation to form the questions. These are more teacher-guided in format, but have lots of opportunities for student input. These stories guide students to expand their thinking to include more possible questions and answer with specifics. Students learn to think about the dimensions of a problem or attributes of an object as they problem solve.

Students do like variety in the problem solving they do. They like story problems that involve familiar people, places or things. Meaningful text is an appealing part of what we term “Aunt Mathilda” problems, a valuable source of stories for developing reasoning (Andrews, 1996). These present situations with an embedded problem to solve and can be found on the internet (http://www.dcmrats.org/auntymath.html). Our students organize the story details so as to know the key question to answer and the information given. We like these Aunt Math problems because the story setting is like a conversation students might hear. Solving the problems from “Aunt Mathilda” might involve classic problems, ones with fractions, or combination problems, written into a story of interest to elementary age students, where the conditions must be extracted from the text.

For example, after reading "Gina's Pet Challenge-Challenge 31" students ponder how many cats and birds are in Gina's big family:

"In my big family, we have 7 pets. Some are cats. The rest are birds. If you count the legs, you will count 20 legs. How many cats and birds do we have in our big family?"

Students read the story until they have a sense of the problem to solve and then how to do it. We encourage them to record what they know, then read for a sense of the problem to solve, then read looking for any special twists that make this problem unique. Students learn to record information step-by-step as they read the problem the second or third time through. In discussing, "What are you trying to find out?" and “Are there any special rules and conditions?” the need to use strategies such as the acting-it-out strategy, manipulatives, pictures, or using charts teaches students ways to organize what is known. The instructor gives guidance, but it is the student who does the thinking, drawing, and sometimes the writing, depending on circumstances. It is stressed that students should work in such a way that they can “prove” their answer is correct and that their thinking is sensible. After solving the problem, students are asked to substantiate their answers as well.

For example, in the Pet Challenge problem, students recorded what they knew and developed a plan. Most identified that the
number of animals was key information and kept track of the head count. Some made 7 birds, and then noted the need for more legs when checking to see if the solution met the requirements of the problem. Some made 7 cats and then realized there were too many legs to fit the conditions, and added birds to modify. A couple of children randomly drew cats and birds and kept a running total of the legs and heads. It was as if they were looking at the interactions of cats and birds as part of the criteria for the problem. Some children used stick figures, some use elaborate drawings and some made a chart showing combinations of heads and legs of birds and cats. Students who have checked their answer against the conditions of the problem seem more assured and confident about their work.

Afterwards, sometimes students ask, "Are there other problems like this we could solve?" Learning to make up problems for each other becomes the next step. Students self-extend the lesson, become more independent learners and refine within themselves the "research skills" of mathematicians as they record ideas, give evidence for their views, and find new information. In solving each other's problems, students note that sometimes they cannot really solve their friend's problem without more information or making an assumption. Furthermore, students’ reading comprehension on word problems seems to improve.

A student teacher observing the students remarked at how the group's discourse was so different from what she would have anticipated. As the students compared their answers, looks of reflection and "hmm" sounds were observed. Instead of hiding the answer to the problem because it looked different, the student teacher noticed students asking each other, "How did you get that?" or "Why do you think so?" The observations surprised her because her past experience included students getting defensive if their answer did not look like their peers. She noted a sense of organization and purpose within their discussions.

Interdisciplinary Tasks

Students also come to enrichment to do interdisciplinary tasks which involve concepts related to the classroom curriculum. Science, social studies and mathematics can be combined as a situation unfolds. We have found that the TIMS (Teaching Integrated Math and Science) and AIMS (Activities Integrating Mathematics and Science) materials are good sources. Students have a question to answer, such as, Which ball bounces the highest when dropped?

Our third graders explored the regularity of Native American designs. For example, blanket border patterns by graphing the number of stencils versus the length of the border covered by the stencils. Materials from TIMS (Beissinger and Kelso, 1995), and the NCTM Math Notes (Barkley, 1997) were combined for the project. Stencils of the design are made with graph paper. Studies of Native American designs in social studies allowed students to come up with some authentic looking patterns. After applying the stencil to a brown paper bag, students measure the length of the pattern after using the stencil once, twice, three times, etc. Student work is taped together to make paper blankets. Looking at the stencil and the resulting design made using the stencil, small groups form hypotheses about the relationship between the length of the design, the length of the stencil and the number of times the stencil pattern is repeated. Students then graph the relationship between the stencil and the design on the brown paper to check their hypothesis. This can be quite exciting. Using the journal, or a
group log on chart paper, students record their understanding of the connection of the stencil length and total length of the design. "Why did it do this?" students want to know. Several students then noticed the connection between the numbers on the Y-axis of the graph and the table. "What are you trying to find out?" took on a new meaning with connection between the numbers and the line. The relationship students see from a repeated pattern and resulting lengths became graphic—it formed a line on the graph. Looking at each other’s data, students are able to make sense of what might be creating the line – it was the consistency of the added amount. Students are able to determine that without the consistency of the repeated addition, the line will not form. Students made a new connection between graphs, repeated addition and multiplication.

### Imponderables

Imponderable problems are the last focus for enrichment lessons in this article. Students are asked to think about problems that are often considered classics. For example, we used the classic boatman across the river problem:

**The Showman, His Tiger, a Duck, and the Sack of Corn**

"How good are you at thinking logically? This is an old problem. You need to help the showman cross a river with his tiger, duck and sack of corn. There is a boat but it is so small that it can only hold the showman and one of the others. The showman can't leave the tiger with the duck as the duck will be eaten. He can't leave the duck with the corn for fear of losing all the corn. How does he get across?" (King, 1999)

Students were asked how to solve this classic logic problem. They were initially provided with a cardboard boat, a blue paper river and puppets. When pairs of students began working on solutions, they used paper pictures of these things. Asking students, "What do you know about the problem?" made students reread to determine the conditions for safe transport. The quick thinkers immediately wanted to have the boat return to the other shore without anyone rowing it, but others pointed out the lack of logic to the idea. Once the fact that there might only be one item brought across at a time, students began working on possible orders for the tiger, duck and corn. In asking, "What is really needed to solve the problem?" another key point became apparent: there might have to be animals returning to the initial shore to keep the three items in safety—the goal was to get all items to the other shore at the end of the boat ride – not just get them to the other side and not think about them anymore. A key point is finally clarified: what is the meaning of "can't leave" in the problem? One pair of students decided that if the showman is part of the group, the duck won't eat the corn nor will the tiger eat the duck. Which item has to go first? After another huddle or two, they now have a solution to share with the rest of the group. Students repeat the conditions of the problem and discuss the logic of their answer. They do this by confirming the meanings of the vocabulary of interest in the problem, such as “can’t leave” and test out the conditions of the problem to check validity of the students' answer. The enrichment aide is available to point out flaws in logic or definitions and help the group know when an acceptable amount of consensus has been reached for that given problem.

Standards for behavior and quality of work are set. The goal is to make the students
responsible for communicating properly, giving the students tools to work together, rather than having the teacher impose a structure. Students cannot put down each other or themselves when stating they do not agree with someone else's answer. The focus is on the facts, not the person. Students are encouraged to take risks in sharing ideas. Mathematicians don't generally have a structure imposed on them when working together; they have to determine what will work for the people involved. Being a successful mathematician includes being flexible in thinking, yet able to verify that conditions of the problem have been met. Students are encouraged to be flexible in their thinking, and encouraged to verify that the conditions of the problem are met before stating they have the correct answer. Extending the problem to a different setting is another way students are thinking like mathematicians, who take solution systems from one setting to another to test validity of a theory and develop new understanding of the setting. Developing reasoning skills in children develops their skill as mathematicians.

References
AIMS Education Foundation, P.O. Box 8120, Fresno, CA 93747-8120


Beissinger, Janet & Kelso, C. (1995). Teaching Multiplication with Patterns, Graphs and Story Writing, presentation at the NCTM Regional Writing Meeting in Chicago IL at Chicago. (More information about TIMS Curriculum Project can be requested at their e-mail address: tims@uic.edu.) TIMS website is http://www.math.uic.edu/IMSE/tims.html


