Students’ Productive Struggle through Mathematical Modeling

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Abstract
This paper is a classroom story in which we present the design and implementation of activities in which model with mathematics, as described in the Common Core State Standards for Mathematics, is a prominent mathematical practice. Using student conversations and written work, we describe the processes that students engaged in and their productive struggles. We then discuss types of understandings and learning exhibited by students through mathematical modeling tasks and present suggestions for a more successful implementation. We conclude with the challenges and benefits of implementing mathematical modeling tasks from the perspective of the teacher.

Keywords: model with mathematics, mathematical modeling, standards for mathematical practice, productive struggle

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With the release of the Common Core State Standards for Mathematics in 2010, Mathematical Modeling (MM) warrants additional attention, given the Common Core’s increasingly important emphasis on real-life applications. Specifically, the Standards for Mathematical Practice connects MM with applying mathematics to solve problems arising in everyday life, society, and the workplace.

The impact of MM on student learning has been studied by many researchers. Lesh and Harel (2003) found that youngsters who were among the least advantaged often invented more powerful ideas than anyone had ever dared to try to teach to them. Lesh and Harel observed processes during modeling that led to extensions, refinements, revisions, or adaptations in students’ ways of thinking. They also found that students were able to make sense of situations based on extensions of their personal knowledge and experiences. Zbiek and Conner (2006) argued that engaging in modeling creates opportunities for conceptual and procedural development for students. They explained that modeling tasks require students to combine multiple properties and parameters into a single mathematical entity, which leads to a deeper conceptual understanding. Furthermore, modeling tasks lead students to use a known procedure in an unfamiliar context or to generate a new or modified procedure, which enhances their procedural understanding. Zbiek and Conner described how learning takes place during MM by defining the following subprocesses of modeling in connection to learning mathematics: exploring, observing mathematically, specifying, mathematizing, combining, analyzing, highlighting, interpreting, examining, and communicating.

Implementing MM tasks often present challenges for teachers. Biembengut and Hein (2010) suggested that teachers’ lack of experience with modeling tasks discourages them from implementing lessons where MM is the core idea. Yildirim, Shuman, and Besterfield-Sacre (2010) found that both time restrictions and the level of guidance (too much or too little) provided by the teacher during students’ solution processes affect the successful implementation of a modeling task. Finally, Gainsburg (2008) suggested that a major barrier for the introduction of modeling tasks is teachers’ fear that their students, due to poor basic mathematical skills, would not engage in solving challenging problems that require critical thinking. Especially in inner-city mathematics classrooms, instruction emphasizes basic skills

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rather than higher levels of understanding (Staples & Truxaw, 2010).

In contrast to the concern expressed by some teachers that the major obstacle to implementing modeling tasks is students’ inadequate mathematical knowledge, the National Council of Teachers of Mathematics (NCTM, 2014) argues that supporting students’ productive struggles while learning mathematics is an effective teaching practice that ensures successful mathematics learning for all. NCTM (2014) describes students’ struggles as opportunities for gaining in-depth understanding of the mathematical structure of problems and the relationships among mathematical ideas. Teaching is no longer defined as providing procedures or as evaluating students’ learning by the correctness of their solutions (Smith, 2000).

In this article, we present the design and implementation of an activity in which mathematical modeling is a prominent mathematical practice. Illustrating with student conversations and written work, we describe the processes that students engaged in and their productive struggles. We then discuss types of understanding and learning exhibited by students during one MM task and present suggestions for a more successful implementation. We conclude with the challenges and benefits of implementing MM from the perspective of the teacher.

1. Modeling with Functions

We partnered with a precalculus teacher from an inner-city Chicago school. The students in the class reported on here were all African-American boys of low socioeconomic status. The teacher developed three MM tasks which ask students to generate functions to describe how one quantity of interest depends on another and to subsequently make decisions. Students were placed randomly into six groups of four and asked to solve two scenarios of their choosing. Each group assigned the following roles to its members:

- **Facilitator/Encourager** keeps the discussion moving, often by asking the other group members questions about what they’ve just been saying.
- **Timekeeper** makes sure that they stay on track and get through a reasonable amount of material in the given time period.
- **Reflector** listens to what others say and explains it back in his own words, asking the original speaker if the interpretation is correct.
- **Recorder/Secretary** takes notes and documents findings.

The students used LiveScribe pens to record their conversations and written work. As a student writes with this pen on dot paper, the pen records the audio and associates it with the written work. Later, conversations about a particular place in the written work may be replayed simply by touching that part of the page with the pen. Students’ conversations and associated written work were downloaded to a computer and transcribed for analysis. In each group, the Recorder/Secretary was given the pen and dot paper to use for note taking.

1.1. Student Conversations and Revealed Understanding

We present conversations among the members of two groups engaged in Scenario A (Figure 1), describe the MM subprocesses that students used, and discuss the mathematical understanding gained by means of productive struggle. We used Zbiek and Conner’s (2006) definitions of subprocesses.

Upon reading the problem, students in Group 1 realized that they had solved similar problems and referred to their notes for help. Clearly, they employed the commonly taught problem-solving strategy of looking for a similar problem. The students started mathematizing as they created a variable \( x \), and they started combining as they blended the givens into a mathematical representation of the problem. Then they were able to analyze the relationship that was represented by the function (Figure 2).

At this point, Group 1 students began to struggle with how to proceed because they were uncertain about whether to determine a value...
Scenario A: Modeling Cost of Text Message Plans
You are choosing between two texting plans. Plan A has a monthly fee of $20 with a charge of $0.05 per text. Plan B has a monthly fee of $5 with a charge of $0.10 per text message.

(a) Express the monthly cost for Plan A, $f$, as a function of the number of text messages in a month, $x$.

(b) Express the monthly cost for Plan B, $g$, as a function of the number of text messages in a month, $x$.

(c) Which plan would your group prefer, and why? Please provide a written statement with your reasoning backed up with calculations.

Figure 1: Scenario A (adapted from Blitzer, 2014)

Student 1: And (a) is: plan A has a monthly fee of 20 dollars with a charge of 5, I mean not 5, but point I mean, I mean 0.05 per text, so I think we gotta write an equation . . . . So probably gonna have to write $a = . . . .$

Student 2: OK. So, a month would be 20 uh OK 5 cents $x$, $x$ times 5 cents plus 20 dollars, right? I think that’s what it would be because . . . . I’m pretty sure that’s what it’s gonna be . . . . so I’m gonna write that down.

Student 1: Wait, yeah, that would make sense and that would be A and then gotta do the same thing for B, so it would be 0.10$x$ plus 5.

Figure 2: Group 1 mathematizing and combining

for $x$ (“I’m trying to find out what $x$ is”) or to choose the best texting plan (“then we gotta find which one is the best price”). Students asked questions related to the sources of their struggles that would help them make progress in understanding and solving the problem. Asking such questions is an indicator of their productive struggle, as described by NCTM (2014). Notice in Figure 2 that these students had not yet explicitly defined the variable, $x$. They defined it later when they started interpreting and examining, i.e., relating the function to real-world conclusions and ensuring that the conclusion aligned with the realistic situation. Specifically, once the students realized that the goal of deciding which plan was better depended on the number of text messages sent in a month, they discussed what would be a reasonable number of text messages sent per month and used their functions to compare the costs of the plans. The discussion led them to the conclusion that Plan A was a better choice. Figure 3 illustrates how students explain their thinking about a task to their peers, which is considered productive struggle by NCTM (2014).

Students in Group 2, on the other hand, first started specifying as they identified and prioritized the key elements of the problem, and mathematizing as they created the variable, $x$, for the number of text messages in a month (Figure 4).

Note an inconsistency between the verbal and symbolic representation for the fee per text message, “every text is a nickel” and “0.5$x$.” Eventually, students noticed and corrected their mistake. However, students began to struggle with specifying because they were sidetracked by the use of the word “month” in the problem (Figure 5). Through further discussion, they came to an understanding that they needed to focus on a month and that the number of days per month did not play a role in modeling the cost: “so, OK, so as of now, what we should do is: for $x$ we should, I mean in my opinion we should, start just randomly plugging in numbers for both Plan A and Plan B to see how much they average out a month.” Figure 5 illustrates students questioning the reasoning of their peers and reflecting on their
Student 2: Just put like this. So, what if ... what if we had like, um, this ... Let’s say the average person sends like 1000 texts a month, more than that right?

Student 1: Uh, I guess you could look at it like that, yeah!

Student 2: Let’s say someone sends 1000 texts a month, right! x is the amount, right? Let’s say, we do it for both. Let’s say 10 cents times 1000 plus 20 ... I mean, plus 5, then you do ...

Student 1: So, you’re just plugging it in, right?

Student 2: Right!

Student 1: So, then, if you do that ... You do 0.10 times 1000 plus 5. You get 105. But, that would be for B.

Student 2: 105. Yeah! That’s B.

Student 1: So, that would be 105, and then, for A ... You would get 70.

Student 2: 70 dollars?

Student 1: Yeah!

Student 2: That’s ... that’s A.

Student 1: That would be right!

Student 2: So, A is gonna be the best choice.

Figure 3: Group 1 interpreting and examining

Student 1: Find monthly cost. They gave us 20 dollars plus .... Every text is a nickel.

Student 2: So, how are we gonna write that as a function? Which is ...

Student 3: Let me think. f of x ... should we write it as f of x? f of x equals ...

Student 1: Then, whatever we got. We got 20 dollars and 5 cents.

Student 3: So, we ... so, we do 20 plus 0.5x.

Figure 4: Group 2 specifying and mathematizing

own understanding through productive struggle.

Both Groups 1 and 2 were able to generate functions to represent the relationship between the number of text messages and the cost of the telephone plans per month (see Figure 6). Group 1, however, showed a higher level of understanding, since they were able to represent the function symbolically, while Group 2 gave only a verbal explanation for how to calculate the cost for each plan. Both groups initially believed that, once a function was written, it needed to be solved for the variable. After realizing that this was not the case, in order to decide which plan was a better choice, each group selected a certain number of text messages a person could send a month and calculated the cost for each plan.

As Lesh and Harel (2003) described in their study, the students in this study were also able to make sense of the situation based on extensions of their personal knowledge and experiences. In particular, Group 1 recommended Plan A based on, in their opinion, a reasonable number of text messages a person may send per month (Figure 6 Group 1 Solution). In contrast, Group 2 concluded that Plan B was a better choice. Those students randomly selected two different numbers of text messages a person could send per month, x = 4 and x = 30, calculated the price of each plan for those values, and found that Plan B was cheaper in both situations. Therefore, they decided that the “per text fee” information was
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| Student 1: f of x equals 20, plus 0.05. |
|--------|-------------------|
| Student 2: But, shouldn’t we take into consideration, month? |
| Student 1: We possibly can, but . . . . |
| Student 2: Let’s just do January. So, January . . . so, a month so that would be 20 plus 0.05x times . . . . |
| Student 3: January got 31 or 30? |
| Student 1: I think 31, but I don’t think we should focus on the days. |
| Student 2: So, just focus . . . I mean, ultimately we can do that for both plan A and plan B. We gonna have to average out the days for a month so . . . I mean . . . It’s not right that we gotta separately add that month. |

Figure 5: Group 2 struggles with specifying extraneous and modified the equation to fit their new understanding of the problem. They made their final decision of which texting plan was best based on monthly charge without considering the per text fee (Figure 6, Group 2 Solution).

2. Discussion

The students did not consider graphing their functions to examine how the cost of each plan depended on the number of text messages per month. Nor did they consider using inequalities to find for which values of x one plan costs more/less than the other. Student conversations and final products revealed that neither group indicated that “best” was relative in this scenario. Students did not realize that the two functions allowed them to compare plans using various numbers of text messages, and they did not provide corresponding recommendations. They modeled two separate functions, but they did not compare them by use of inequalities or graphs, although they were previously exposed to those concepts. This shows that for these students, inequalities and function graphs appear to be concepts that are not “previously internalized and currently accessible” (Zbiek, 1998).

As indicated in the literature, the level of guidance provided by the teacher during students’ solution processes might have affected students’ success on the modeling task (Yildirim et al., 2010). Additional scaffolding might have allowed the students to consider other numbers of text messages per month, to graph each plan and deduce additional information from the graph, or to make connections to inequalities. For example, the teacher could have asked, “Would you change your recommendation for a different number of text messages sent per month? Why or why not? Is there a number of messages for which it does not matter which plan was chosen? Explain.” Such questioning would provide students with opportunities to discuss...
Scenario B: Modeling the Number of Customers and Revenue

On a certain route, an airline carries 6000 passengers per month, each paying $200. A marketing survey indicates that for each $1 increase in the ticket price, the airline will lose 100 passengers.

(a) Express the number of passengers per month, \( N \), as a function of the ticket price, \( x \).

(b) The airline’s monthly revenue for the route is the product of the number of passengers and the ticket price. Express the monthly revenue, \( R \), as a function of the ticket price, \( x \).

(c) Prepare an email to the owner of Urban Airline highlighting one of two recommendations:

- Fare should be increased to what amount and the rationale.
- Fare should not be increased and provide your justification.

Figure 7: Scenario B (adapted from Blitzer, 2014)

Another factor mentioned by Yildirim et al. (2010) that had bearing on student success was time. Students felt pressured by the amount of time given for completing two tasks. Hence, they shortened their discussions regarding Scenario A and moved on to solving a second scenario. A discussion less constrained by time might have resulted in more comprehensive answers.

As previously stated, students were given the opportunity to solve two of the three given scenarios. Scenario A was selected by all groups. They were able to make sense of the problem using personal knowledge and experiences, but only three groups considered functions as a way to complete the task.

In contrast, students had little success with modeling the other two scenarios. The two groups that attempted Scenario B (Figure 7) unsuccessfully tried to generate a revenue function (Figure 8).

Four groups attempted Scenario C (Figure 9). Two of those groups were able to represent the situation appropriately with a diagram and knew that the volume formula was relevant, but were unable to connect the two (Figure 10).

We already mentioned time as one of the factors that might have influenced students’ performance. It is also possible that students had no or little knowledge about revenue or how to make a box, suggesting that students’ familiarity with the context of the problem situation may be relevant. In this case, additional guidance from the teacher (e.g., providing background information on monthly revenue or a box or materials to make a box) could have enhanced students’ understanding of the situation, and eventually their solution of the problem. This is described by NCTM as giving “students access to tools that will support their thinking processes” (2014, p. 49).
Scenario C: Obtaining a Function from a Geometric Formula
A machine produces open boxes using square sheets of metal measuring 12 inches on each side. The machine cuts equal-sized squares from each corner. Then it shapes the metal into an open box by turning up the sides.

(a) Express the volume of the box, \( V \), in cubic inches, as a function of the length of the side of the square cut from each corner, \( x \), in inches.

(b) Find the domain of \( V \).

(c) Prepare a memo to be given to the production manager at Motorola recommending what size squares should be cut and why.

To conclude, students’ productive struggles during mathematical modeling (MM) provided several opportunities for both procedural and conceptual development. Although students were not able to use their models to come up with a comprehensive answer, they challenged their previous understandings of functions by means of MM. Learning occurred within various subprocesses of modeling as defined by Zbiek and Conner (2006). For example, during specifying, students developed mathematical insight as they were trying to elicit the important parameters. During interpreting, they came to realize that \( x \) is a variable that may take various values, and they started assigning values to \( x \). An important moment happened when they differentiated between solving an equation versus evaluating a function. They conceptualized modeling as formulating a relationship between entities, i.e., creating a function. They made new connections between their knowledge of two mathematical objects—equations and functions—and came to realize that a function expresses the relationship between variables whereas an equation (with one variable) states a condition on a single variable. Lastly, analyzing naturally led to opportunities to improve procedural understandings.

Hence, students’ struggles though mathematical modeling were opportunities to learn. Moreover, it was a valuable experience for the teacher. NCTM (2014) recommends that mathematics classrooms that embrace productive struggle necessitate rethinking on the part of teachers what it means to be an effective teacher of mathematics. In this regard, in Figure 11 the teacher reflects on the challenges/benefits of implementing a modeling lesson, what his students learned during the process, and how this experience influenced him to rethink his teaching practice.
Teacher’s Reflection

One difficulty in getting the project together was the creation of the material with the scenarios and the document with explicit descriptions of the various roles of group members. A major issue surfaces when we recognize that there is a seeming disconnect between the students’ understanding of what’s being asked and their ability to translate it appropriately into mathematical symbols. Additionally, even when students are able to do the aforementioned items, they can perform stellar computations but have a hard time interpreting what the results mean in the context of the given problem. Even though these are valid concerns, I walked away from the project with a deep appreciation for the struggle involved and understood that this is, in essence, an opportunity for students to make sense of problems and to persevere in solving them.

Another aspect of the project that became more apparent to me was the importance of putting thought into how groups are assigned. I wanted students to work with others outside of their normal circle. Therefore, I shuffled a deck of cards and randomly issued students a card upon entering the classroom. Once all students were in the room, I announced that the groups would be made up of those who had the same number regardless of the suit. This gave me a great mix of students. As the lesson unfolded, I began to wonder how it could have been done differently had higher level students been paired up with lower performing students to stimulate more conversation and to expose the students to various levels of mathematical thinking.

The implementation of the project was fairly smooth. I took a few moments at the outset to ensure that students understood everything that was included in their packages and understood their roles. It appeared that students were well aware of what was supposed to happen during the course of their groups. Since the LiveScribe pen was recording, it appeared that students were more task-oriented than usual. I learned that when students understand the expectations, all components are laid out properly, and there are concrete deliverables required at the conclusion of the task, they are more engaged and more committed to the task at hand.

When walking around and looking at the various ways students approached the assignment, I gained a deep appreciation for the value students can bring to a lesson. Even when students approached the assignment in ways that I could not fathom giving a proper result, it was interesting how other students posed clarifying and probing questions designed to critique and challenge the reasoning of their peers. This helped me to think more carefully about the shift required for our students to become mathematically literate individuals in our changing society. I walked away from this experience seeing hands-on in real time that students are more than capable of carrying out these roles when afforded the opportunity to do so. Moreover, I learned that students want to talk and be engaged in mathematical discussion but in many instances they cannot get a word in edgewise because of the structure in the traditional classroom. Participation in this project has made me re-think my entire philosophy of teaching mathematics, and led me to redesign and restructure my desk/table set-up in a way where conversation is fostered. I have opted for a two-level U-shaped arrangement, and it has impacted students’ conversation and productivity in ways I could not imagine.
References


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